Non-standard gap

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Here's a game, entitled « non-standard gap », which lets you figure out if the person you're speaking to has a non-standard mind, or - and surely it's the same thing - if they're secretely a set theorist.

This is a two-player game where Alice and Bob play in turns. At the $(n+1)$-th round, Alice sets forth[1] an ordinal $\alpha_n > 0$, then Bob proposes an ordinal β_n, with the following constraints:

$$\alpha_0 > \alpha_1 > \cdots > \alpha_n > \cdots \geqslant \beta_n \geqslant \cdots \geqslant \beta_1 \geqslant \beta_0. \tag{1}$$

Alice loses the game in the n-th round if Bob submits her a winning strategy, with proof that in less than a determined number $f(n) > 0$ of rounds after this n-th one, Alice won't be able to extend her sequence while observing the constraints[2]. Bob loses the game if he gives up or dies of thirst.

Example. Alice plays the number ω^ω;
Bob plays ω^2;
Alice plays ω^{10000};
Bob plays $\omega^{9999} + 1$;
Alice plays $\omega^{9999} + \omega^{9998}$;
Bob plays $\omega^{9999} + \omega^{9997} + \omega^{9996}$;
Alice plays $\omega^{9999} + \omega^{9997}\, 2 + \omega^{9996}\, 3 + \omega^{9995}\, 4 + \cdots + \omega^{1221}\, 8778$.
Bob then proposes the following winning strategy:
If Alice has just played the number

$$\alpha_n = \sum_{k \leqslant 9999} \omega^k\, a_n(k)$$

where $a_n \in \mathbb{N}^{10000}$, then Bob will play

$$\beta_n := \sum_{k \leqslant 9999} \omega^k\, b_n(k),$$

where b_n is the predecessor of a_n in $[\![0, \max\{a_n(k) : k \leqslant 9999\}]\!]^{10000}$ for the appropriate lexicographical ordering. Bob claims, with proof, that Alice will lose in at most 1222 additional rounds.

Question 1. Does Bob have a winning strategy? The ordinal α_0 being fixed, does Bob have a winning strategy? If not, then which is this ordinal α_0 that is smallest, for which Bob has no winning strategy?

Question 2. Does Alice have a winning strategy?

Remark. I never lost[3] a game of non-standard gap.

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1. defines
2. that is, we'll have $\alpha_{n+k} = \beta_{n+k-1} + 1$ for a certain $k \in \{1, \ldots, f(n)\}$
3. played